World graph formalism for Feynman amplitudes

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ABSTRACT: A unified treatment of Schwinger parametrised Feynman amplitudes is suggested which addresses vertices of arbitrary order on the same footing as propagators. Contributions from distinct diagrams are organised collectively. The scheme is based on the continuous graph Laplacian. The analogy to a classical statistical diffusion system of vector charges on the graph is explored.

Contents

1. Introduction

Recently, the well-known technique of Schwinger parametrisation of Feynman diagrams [11] has received renewed interest, funneled by the speculation of Gopakumar [8 – 10] that it might be the key to an understanding of the AdS/CFT correspondence on the diagrammatic level of correlation functions. This suggestion is based on the observation that the "lifting" of free large N U(N) symmetric gauge field theory amplitudes of twist-2 operators from the boundary into bulk AdS space has a very natural appearance when one applies the Schwinger parametrisation to the boundary amplitudes, at least in the simplest nontrivial case of three-point amplitudes. The particular integral representation of the bulk amplitudes obtained in this manner is being interpreted as a string theory on a curved space in the limit of large curvature; and since such a theory is currently beyond a direct understanding, these results consequently incited a program of trying to gain knowledge about this particular string theory by the study of the lifted boundary field theory. According

to the advertised model, the correspondence proceeds in two clearly distinct levels: First, there should be a correspondence between the boundary amplitudes and an open string theory including branes; second, by open-closed duality, these open string amplitudes should be equivalent to closed string amplitudes living in the bulk (see e.g. [15]).

In this paper, we concentrate on the Schwinger parametrisation of quantum field theories. In its simplest form, it is obtained by going to the momentum space representation of a Feynman diagram derived from the path integral and rewriting it, making use of the representation

$$\frac{1}{q^2 + m^2} = \int_0^\infty \mathrm{d}\tau \, e^{-\tau(q^2+m^2)}. \tag{1.1}$$

The issue that Schwinger parametrisation can be interpreted as being generated by "world line" path integrals is rather settled by now [19, 20]. In section 8 of Schubert's review [19], the question of how to treat multi loop Feynman graphs in that context is discussed, noting that the world line formalism cannot be implemented immediately on those graphs since — in opposition to one-loop graphs — the graph cannot be treated as a differentiable manifold (parametrised S^1), due to the vertices. The solution offered is rather a pragmatic one: Multi-loop Feynman graphs are constructed from a one-loop "spider" graph with several external insertions, by connecting some of the external insertions by propagators in Schwinger parametrisation. The resulting amplitudes are then manipulated algebraically and the loop momenta are integrated out.

It is the intention of this text to suggest a direct, stringent procedure implementing the world line formalism also for graphs with vertices, without resorting to iterative construction out of simpler graphs. We show how a multi loop Feynman graph can be treated as one-dimensional manifold with branching points, enabling us to write down a "world graph" formalism which delivers equivalent results (e.g., formula (4.11) on page 16) in a consistent, "one-step" fashion. It takes the form of a simple diffusion path integral mapping the *complete* Feynman graph into coordinate space, with particular continuity conditions at the vertices. The main result reported in this article is the methodical derivation of this new route. A side result is to offer a more detailed view on the role of generalised Schwinger parameters, or "moduli", of Feynman graphs.

The formalism is not restricted to a particular set of vertices, or particle types (although it is rather natural to employ it for massless particles). From this point of view, Schwinger parametrisation is a notion which makes sense for a Feynman graph as a whole - one should rather speak of "world graphs" than world lines. We will show there is a close connection to the interpretation of Feynman amplitudes as a partition sum of charged particles residing on the graph, generalising a concept which has successfully been applied to one-loop and two-loop graphs. It is crucial that these partition sums are in fact sums over all different possibilities to connect the external propagators to the graph. Polynomial prefactors in the internal momenta of the Feynman amplitudes, i.e. derivative interactions, for vector or tensor particles can be included easily by introducing infinitesimal "test-dipoles" on the graph.

Let us mention at this point the connection to string theory: Bern and Kosower [3] have shown in a long work that Schwinger parametrised amplitudes can be obtained from the infinite tension limit of a certain open string theory, where the strings degenerate into

point particles. The tachyonic modes of the string can be made to decouple, and the only excitations left in this limit are the massless modes (all other modes become too massive to be excited at all). The Feynman rules which result in this limit come out very naturally in the Schwinger parametrised form (however see e.g. [17] for a different limit retaining only the *tachyonic* modes, producing scalar ϕ^3 theory). In fact, we can say more: The theory of the massless vector fields obtained in this way is a Yang-Mills theory; if the strings carry Chan-Paton factors, then it is a non-Abelian gauge theory [16]. Now, in the usual Feynman diagrammatic calculation of amplitudes in non-Abelian gauge theories, there is a lot of redundancy: The amplitudes corresponding to the diagrams consist of *very* many different summands, and there occurs a host of cancellations between those, so that the final result usually reduces to a comparably compact expression. When the same amplitudes are derived by way of the infinite-tension limit of string theory, they turn out to be very well organised so that cancellations are immediate [2, 14].

Let us mention another interesting detail: On the string theory side, we have to integrate over the so-called "string moduli". These are parameters which label uniquely the conformally inequivalent ways to put a metric on the string world sheet. Taking the infinite-tension limit, the string moduli are mapped partially onto the Schwinger parameters. A point we want to stress is that the mathematical problems which are a major obstruction when one tries to consider more complicated string world sheet topologies in the infinite tension limit are understood rather naturally in the world graph limit.

We thank the referee for pointing out to us an earlier work by Dai and Siegel [4], who explore a related approach to multi-loop amplitudes. As a starting point, they choose the first-quantised formalism, developed by Strassler [20] and many others, which includes an integral over the reparametrisation group of the parametrised Feynman graphs, and requires the subsequent fixing of this reparametrisation symmetry. Their conclusions are similar, stressing along the way the "electrical analogy" which is obtained when the momentum flux through the diagram is set in analogy to a (vector) current. Explicitly, their approach is spelled out only for scalar fields.

The organisation of this paper is the following: In section 2, we introduce the naïve Schwinger representation and show how for each propagator it can be interpreted as a diffusion kernel, implying the world line picture. In section 3, we introduce the world graph formalism, enlarging the diffusion scheme to complex graphs, and state its main content as a theorem. In section 4, we show the equivalence of the world graph scheme to the partition sum of a system of charged particles residing on the graph and complete the proof of the theorem; we give some elementary examples of the technique. Finally, in the remaining section, we extend the formalism to vector and tensor particles.

2. World-line formalism

2.1 Schwinger parametrisation

The Schwinger parametrisation of the correlation functions of a Lagrangian field theory in d-dimensional Euclidean space containing a set of scalar fields and an arbitrary non-derivative polynomial interaction is based on the perturbative expansion of the effective
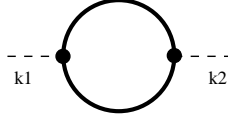
Figure 1: Self energy diagram of ϕ^3-theory. Dashed lines: amputated legs. Momenta are incoming.

action in momentum space. The effective action is the sum over connected, amputated Feynman diagrams, containing massive propagators

$$G_m(q) = \frac{1}{q^2 + m^2} \tag{2.1}$$

and vertices with varying coordination number n, carrying a momentum conserving factor $-\frac{c_n}{(2\pi)^{(\frac{n}{2}-1)d}} \delta^{(d)}\left(\sum_{j=1}^n q_j\right)$, where c_n is the coupling and q_j are the incoming momenta.

To a vertex v, we assign the external momentum k_v, the total sum of the joint momenta entering the diagram through *all* external, amputated legs of v. External legs are thus effectively represented by vertices with non-conservation of momentum of the internal propagators. Conversely, internal vertices will be treated as being connected to imagined external legs with zero momentum entering the graph. Finally, the internal momenta are integrated over.

The naïve Schwinger representation is obtained by blindly representing each internal propagator (2.1) by formula (1.1). We have thus for each propagator j a Schwinger modulus (Schwinger parameter) τ_j. As a result, loop momenta integrations are Gaussian and can be performed explicitly, leaving only the integrals over the Schwinger parameters. The result is the well-known formula which reduces to a certain sum of "two-trees" of the graph (e.g. [13]); the precise form is irrelevant here.

Formula (1.1) has an interpretation based on the diffusion equation. Each vertex v obtains an additional coordinate $x_v \in \mathbb{R}^d$; momentum conservation at v is represented by the integral $\delta^{(d)}\left(\sum_{j=1}^n q_j\right) = (2\pi)^{-d} \int d^d x_v \exp - \left(i x_v \cdot \sum_{j=1}^n q_j\right)$. A Schwinger parametrised propagator running from vertex x_1 to vertex x_2 evaluates to

$$
\begin{aligned}
G_m(x_1 - x_2) &= \int_0^\infty d\tau \int d^d q \, \exp\left(-i(x_1 - x_2) \cdot q - \tau(q^2 + m^2)\right) \\
&= \pi^{d/2} \int_0^\infty d\tau \, \frac{1}{\tau^{d/2}} \exp - \left(\frac{(x_1 - x_2)^2}{4\tau} + m^2 \tau\right). \quad (2.2)
\end{aligned}
$$

If there are external legs carrying momentum k_v attached to vertex v, then we are left with the factor

$$\exp - (i x_v \cdot k_v).$$

As an example, the one-loop self energy of scalar ϕ^3 theory (cf. figure 1) is

$$
\begin{aligned}
I(k_1, k_2) = {} & \frac{1}{2} \cdot \frac{c_3}{(2\pi)^{\frac{3d}{2}}} \int d^d x \, \exp\left(-i x \cdot k_1\right) \cdot \frac{c_3}{(2\pi)^{\frac{3d}{2}}} \int d^d y \, \exp\left(-i y \cdot k_2\right) \\
& \pi^{d/2} \int_0^\infty d\tau_1 \frac{1}{\tau_1^{d/2}} \exp - \left(\frac{(x-y)^2}{4\tau_1} + m^2 \tau_1\right) \\
& \pi^{d/2} \int_0^\infty d\tau_2 \frac{1}{\tau_2^{d/2}} \exp - \left(\frac{(x-y)^2}{4\tau_2} + m^2 \tau_2\right).
\end{aligned} \tag{2.3}
$$

By shifting $y \to y + x$, we eliminate x from the quadratic exponent; the x-integration is then seen to represent external momentum conservation.

The propagator (2.2) is related to the well-known Gaussian kernel for the Wiener path integral describing the diffusion of a particle for a time τ in d (Euclidean) dimensions. We can write it formally as

$$
\begin{aligned}
\mathscr{K}_\tau(x_1, x_0) &= Z_0^{-1} \int_{\substack{x(\alpha)=x_1 \\ x(0)=x_0}} \mathscr{D}x(t) \exp - \int_0^\alpha \mathrm{d}t \left\{ \frac{1}{4} \dot{x}^2 \right\} \\
&= \left(\frac{1}{4\pi\alpha} \right)^{d/2} \exp - \frac{(x_1 - x_0)^2}{4\tau}.
\end{aligned}
\tag{2.4}
$$

The prefactor $\frac{1}{4}$ in the exponent is chosen in order to agree with the standard literature, e.g. [20]. The kernel is normalised to

$$
\int \mathrm{d}^d x_1 \, \mathscr{K}_\tau(x_1, x_0) = 1.
\tag{2.5}
$$

Comparing with (2.2), we find that the propagator for a scalar quantum field can be expressed through the heat kernel by

$$
\begin{aligned}
G_m(x - y) &= \pi^{d/2} \int_0^\infty \mathrm{d}\tau \, \frac{1}{\tau^{d/2}} \exp - \left(\frac{(x-y)^2}{4\tau} + m^2\tau \right) \\
&= (2\pi)^d \int_0^\infty \mathrm{d}\tau \, \exp\left(-m^2\tau \right) \mathscr{K}_\tau(x - y).
\end{aligned}
\tag{2.6}
$$

One could interpret the exponential prefactor as a dissipative term absorbing the diffusing particle. The particle mass m enters only in the dissipative part, and for $m = 0$, we are left with the dissipation-free heat kernel. This formula demands $[\tau] = L^2$, so we must be cautious when we interpret τ as "time" (the reason is that we have left out the diffusion constant). The upshot is that the Euclidean field theory propagator is obtained by averaging over diffusion "times" τ, with a weight factor falling off exponentially. The diffusion picture does not extend to cover vertices, as it is.

In this approach, the Schwinger parametrised form of the correlation function reads for any mass

$$
\begin{aligned}
G_{\mathscr{G}}(k_1, \ldots, k_n) = \frac{(2\pi)^{\frac{nd}{2}}}{\mathrm{Sym}(\mathscr{G})} \Big(\prod_{\text{vertices } v} -c_v \int \mathrm{d}^d x_v \, e^{-ik_v \cdot x_v} \Big) \\
\prod_{\text{propagators } j} \int_0^\infty \frac{\mathrm{d}\tau_j}{(4\pi\tau_j)^{d/2}} \exp - \left(\frac{(x_{1(j)} - x_{2(j)})^2}{4\tau_j} + m^2\tau_j \right).
\end{aligned}
\tag{2.7}
$$

Here, $\mathrm{Sym}(\mathscr{G})$ is the symmetry factor of the graph $\mathscr{G}$, and all powers of 2π at the vertices have been cancelled against the propagators.

2.2 Conformal propagators

The Schwinger parametrisation is useful also to represent conformal propagators

$$
G_\Delta(x - y) = \frac{1}{|x - y|^{2\Delta}}.
\tag{2.8}
$$

The scaling behaviour will be contained solely in a τ-dependent prefactor. Introducing a Schwinger-like integral representation in coordinate space

$$\frac{1}{(x^2)^\Delta} = \frac{1}{\Gamma(\Delta)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{\Delta-1} e^{-\alpha x^2}, \qquad \Re\Delta > 0,$$

we can compute the Fourier transform as

$$\int \mathrm{d}^d x \, e^{-iq\cdot x} \frac{1}{(x^2)^\Delta} = \frac{1}{\Gamma(\Delta)} \int \mathrm{d}^d x \, e^{-iq\cdot x} \int_0^\infty \mathrm{d}\alpha \, \alpha^{\Delta-1} e^{-\alpha x^2}$$
$$= \frac{\pi^{\frac{d}{2}}}{\Gamma(\Delta)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{\Delta-\frac{d}{2}-1} e^{-\frac{q^2}{4\alpha}}$$

by completing the square. Substituting $\alpha \to (4\tau)^{-1}$, we get the usual Schwinger parametrisation

$$G_\Delta(q) = \frac{2^{d-2\Delta} \pi^{\frac{d}{2}}}{\Gamma(\Delta)} \int_0^\infty \mathrm{d}\tau \, \tau^{\frac{d}{2}-\Delta-1} e^{-\tau q^2}. \tag{2.9}$$

This representation is special insofar as the exponential part takes exactly the form of a massless propagator. The only modification is the power of τ in the Schwinger kernel. If $\Re\Delta < \frac{d}{2}$, we can evaluate the integral explicitly to obtain

$$G_\Delta(q) = \frac{2^{d-2\Delta} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)} |q|^{2\Delta-d}. \tag{2.10}$$

Note that even when Δ is not within the bounds indicated, we may by analytic continuation reach almost every complex Δ.

3. World-graph formalism

Bosonic string theory can be formulated as a theory of d scalar "coordinate" fields living on the two-dimensional string world sheet (a Riemann surface which can have an arbitrary topology). In the infinite string tension limit, strings are effectively reduced to point-like particles (string shrunk to zero length). Only very few string excitations survive this limit, and it has been shown that it can be consistently treated as a field theory.

The Feynman graphs of the limiting field theory may be quite literally interpreted as the shrivelled remains of the string world sheet under infinite tension. As they are one-dimensional, it has become customary to refer to the propagators in this context as "world lines"; they are sewn together at the vertices. On the technical level, scattering amplitudes mediated by string interactions are turned into a Schwinger parametrised version of field theoretic perturbation theory. From a world sheet point of view however, the discrimination between "free strings" and "vertices" is artificial; given a section of the world sheet, the question of whether it is a part of a "vertex" or not does not make sense at all.

Let us reverse the argument and ask whether there is an approach to field theory which resolves the special treatment of the vertices. This approach is found in the generalisation of Schwinger parametrisation. Instead of using "world lines", we will employ the concept of a "world graph" — a Feynman graph $\mathscr{G}$ is treated as a manifold with branching points
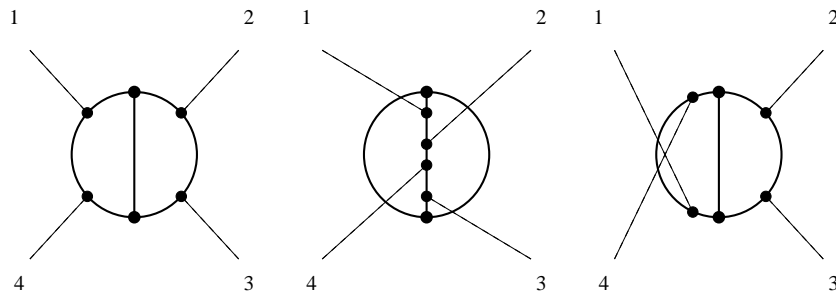
Figure 2: Three different graphs grouping into the same equivalence class under class scheme A.

at the vertices which is mapped into the ambient space $\mathbb{R}^d$. The *world graph path integral* is a weighted integral over all allowed embeddings of this kind. No longer enter external particles the world graph at "special points" (vertices); rather, they are implemented by sliding "operator insertions" which have to be taken care of in the path integral.

At the branching points (vertices) of the world graph, it will be required to impose a continuity condition. It is precisely this continuity condition which marks the difference between world line and world graph formalism. The representation of the propagators is taken over unmodified from the world line formalism: We associate a "length" τ_j to each propagator in the graph; the final amplitude is obtained by performing the world graph path-integral and integrating over all lengths with the appropriate weight factor. The lengths τ_j will be called "moduli", in reference to the term used in string theory, where the moduli characterise different conformal equivalence classes of Riemannian metrics on the string world sheet. The parameter space for the moduli is the *moduli space* $\mathrm{Mod}[\mathscr{G}]$. The "dimension" $\dim[\mathscr{G}]$ of a graph $\mathscr{G}$ is by definition the dimension of its moduli space $\mathrm{Mod}[\mathscr{G}]$. It is important to recognize that the positions t_j of the operator insertions are part of the moduli. Graphs which are assigned such Schwinger lengths will be called "metric", in distinction to the usual Feynman graphs without Schwinger lengths, which will be called "non-metric".

3.1 Equivalence classes and cells of moduli space

In the usual Schwinger parametrised Feynman graphs (world line formalism), the order of the external (amputated) legs entering the graph is fixed. The integration of the Schwinger parameters varies just the distances in-between them; this is equivalent to letting the insertions slide over the branches of the graph, without changing their order, and integrating over all possible branch lengths.

The world graph formalism does not admit a special treatment of vertices; so we should expect that it makes sense to integrate over *all* admissible localisation points and orders of the insertions. When we integrate over moduli space and let the operator insertions slide over the diagram, we find a natural sum over different orderings of the external legs on the same branch of the diagram, and also include the case where the external legs are inserted onto different branches (see figure 2). This clearly means that using the world graph formalism, we necessarily will obtain *sums over different Feynman graphs*. Each sum defines a subset of the total set of all admissible Feynman graphs (where we accept

the Feynman rules as a priori given), and because the world graph formalism should be equivalent to the usual sum over Feynman graphs, it is important that each graph is part of precisely one such sum. In other words, the set of all Feynman graphs is partitioned into (mutually exclusive) equivalence classes. The class containing the connected graph $\mathscr{G}$ will be denoted $[\mathscr{G}]$; it is a set of non-metric connected Feynman graphs. The moduli space cells $\mathrm{Mod}[\mathscr{G}]$ are in fact parametrising these equivalence classes, rather than just single graphs. Given a class $\mathfrak{g} \equiv [\mathscr{G}]$ and a particular set of moduli $\tau \in \mathrm{Mod}\,\mathfrak{g}$, the particular metric graph identified by the moduli will be denoted $\mathfrak{g}(\tau)$. The moduli space $\mathrm{Mod}\,\mathfrak{g}$ is a measure space of dimension $\dim\mathfrak{g}$, and the moduli τ are coordinates on this space. The measure on $\mathrm{Mod}\,\mathfrak{g}$ is derived from the proposed equivalence to the usual diagrammatic computation.

We will suggest a set of rules telling us how, given a graph $\mathscr{G}$, we can subsequently generate all $\mathscr{G}' \in [\mathscr{G}]$ in the equivalence class; it is easy to show that these relations are reflexive, symmetric and transitive, and therefore define a true partitioning into equivalence classes. Let us point out that there are several consistent ways to define the classes; we discuss them in turn.

As a prerequisite, we have to classify the operator insertions: Each insertion is connected to a number of external legs entering the diagram, its "external valency"; in the simplest case, it will be one. We assume that all external legs are distinguishable. Likewise, there is a number of internal legs connected to the insertion (the "internal valency"). The sum of external and internal valency is the total valency of the insertion, and this settles the coupling needed at the insertion (ignoring the question of different particle types for the moment). *We stress once more that "ordinary" vertices are treated throughout as operator insertions with external valency 0 in the world graph formalism.*

Definition (Equivalence Class A). Given a graph $\mathscr{G}$, the equivalence class $[\mathscr{G}]$ is generated by letting insertions with internal valency 2 slide over the complete graph, changing their order as they go along. Insertions with internal valency other then 2 (1 or larger than 2) cannot slide; however, if there are several insertions with identical valencies (internal *and* external), then the external legs attached to these insertions may be *permuted* groupwise; i.e. if several external legs are attached to the same insertion in $\mathscr{G}$, then they must be attached to the same insertion for every graph in $[\mathscr{G}]$ (external legs are "sticky").

The continuous moduli are thus given by coordinates of the sliding insertions on the graph, and by the metric of the underlying "torso";[1] in addition, there are discrete moduli counting possible permutations of the external legs.

This definition may seem a little arbitrary; however, it is the one which is best suited to the statistical analogy which will be introduced below. An example of discrete moduli is given in figure 3. We will nevertheless write the integral over moduli space as $\int \mathrm{d}^{\dim[\mathscr{G}]}\tau$.

There are further possibilities: A rather strict one is

Definition (Equivalence Class B). Given a graph $\mathscr{G}$, the equivalence class $[\mathscr{G}] = \{\mathscr{G}\}$ is minimal. There are no permutations or rearrangements. The corresponding moduli space $\mathrm{Mod}[\mathscr{G}]$ is made up of the usual Schwinger parameters.

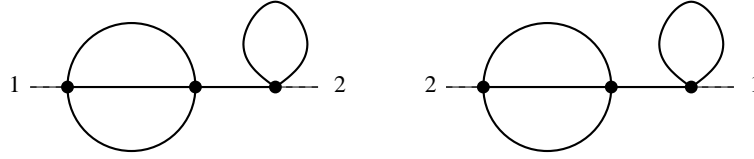[1]Symmetries will be discussed below.

Figure 3: External legs 1 and 2 can be interchanged without modifying the valency of the insertions. There is a discrete modulus (eg $\tau_{12} \in \{12, 21\}$).

This is the class concept of the world line formalism. Finally, we may be very liberal and make

Definition (Equivalence Class C)**.** Given a graph $\mathcal{G}$, the equivalence class $[\mathcal{G}]$ contains all graphs which can be assembled by cutting all internal propagators of $\mathcal{G}$ and reconnecting the remaining insertions (retaining their internal valency) in an arbitrary manner, under the constraint that the resulting graph is connected.

The moduli space counts all different possible topologies, and on each topology there are continuous moduli controlling the metrics.

Note that this includes "discrete permutations"; the external legs are still sticky. It is easy to see that the loop number is constant within a class; the remaining class invariants are derived from the types of operator insertions. This latter class concept is the one which is most closely related to closed string theory: A closed string world sheet with finite tension is topologically characterised solely by its loop number. Incidentally, it is the one which is obtained following the approach of Bern and Kosower [3].

In a wider sense, the moduli space of all possible graphs is made up of different "cells" $\text{Mod}[\mathcal{G}]$. In this way, the total moduli space (containing all graphs) has a natural cell structure [10]. The class concepts introduced regulate the extent of these cells. It is obvious that $B \leq A \leq C$, i.e. B defines a subpartitioning of A, and A subpartitions C. The partitioning A seems to be the one which incorporates systematically the world graph concept without making the classes unnecessarily large. This should not be mistaken for a physical statement: It is rather one of convenience.

On the other hand, the partitioning C is the only one treating internal and external valencies alike. This can be seen by following example: Consider two vertices v and w sliding along one branch of a graph $\mathcal{G}$; assume that these vertices are distinguishable (ie by their total valency); for concreteness, assume that vertex v is 3-valent and vertex w 4-valent. We have to decide what to do with the one remaining leg of v and the two remaining legs of w. If the remaining legs of the vertices are all external (and thus amputated), then v and w may exchange their order on the branch within $[\mathcal{G}]$ by A. However, if the remaining legs are connected internally by a propagator to some other vertex of the graph, they may not, by A. This is not so in C: There, diagrams are completely rewired, and every ordering is included.

For the purpose of the examples given in this text, the concept A is broad enough. The generalisations to C are immediate in most cases. For this reason, *we will in the rest of this text adhere to the class concept A.*

Symmetry factors. Each Feynman graph has to be divided by a symmetry factor which is obtained in the usual way from the perturbative expansion (it is the size of the automorphism group of the graph). These factors are identically taken over in the world graph formalism. If the topology of the underlying diagram varies within a cell of moduli space, the symmetry factor may change. For an important practical aspect, see however the comments at the end of section 3.2.

Particle types. If there are different sorts of particles involved, then every propagator has to be assigned a particle type. Two graphs of the same topology are different by definition if the particle types are not completely identical. The particle types of external, amputated legs are fixed by assumption. In this case, we make the agreement that all possible assignments of particle types to the internal propagators which can be satisfied by a set of vertices from the Feynman rules are part of the class. As we generate the graphs in a class, each time the topology changes or one insertion crosses another, we have to change the particle type of the propagators. This implies that the couplings have to vary as well. There will be topologies that can be fulfilled (because the Feynman graph corresponding to this particular ordering can be constructed from the couplings) and topologies that will fail (because there is no corresponding Feynman graph). There have to be additional, discrete moduli to keep track of particle types and couplings.

As far as we allow arbitrary sets of vertices and particles, this is already the end of the story. When the analogy to string theory is deepened however, we expect that string theory puts serious restrictions on the possible types of particles and vertices, and their coupling constants. The simplification of the amplitudes which has been mentioned in the introduction should be present only for very particular theories, and supposedly the field theories motivated by string theory are strong candidates here. Eg, we expect such simplifications for non-Abelian gauge theories. The choice of a suitable class concept is at the heart of these supposed simplifications.

Although, if we take serious the string theory parallelism, we should only consider massless propagators, massive particles can without problems be included into the scheme, with certain qualifications. If all propagators have the same mass, then the mass prefactor is trivially given by the total length of the graph. If the propagators carry different masses, we will not be able to give such a concise description: The operator insertions describing external legs change the "phase" (mass) of the world graph lines. A possible way out is the inclusion of a further "particle type" or mass field $m(t)$ on the world graph and to describe the operator insertions as symmetric matrices connecting different "mass" spaces (see section 4.6 below).

3.2 Formulation of world-graph path integral

Let us recapitulate: To compute the amplitude corresponding to an equivalence class of Feynman graphs $\mathfrak{g}$, we first select one particular metric graph $\mathfrak{g}(\tau)$ by a choice of Schwinger parameters $\tau \in \mathrm{Mod}\,\mathfrak{g}$. On $\mathfrak{g}(\tau)$, we put a theory of d Euclidean massless scalar fields $x : \mathfrak{g}(\tau) \to \mathbb{R}^d$ whose dynamics is described by a diffusion (Wiener) process. External legs are treated as operator insertions $e^{-ik_v \cdot x(t_v)}$; and after integrating the fluctuations of the d

coordinate fields $x(t)$, we have to integrate over the moduli space Mod $\mathfrak{g}$. This contains an integral over all possible positions of the operator insertions on $\mathfrak{g}(\tau)$ as well as all possible Schwinger lengths of the propagators with the appropriate measure; there is a sum over the discrete moduli, controlling permutations of the external legs, and finally over the different topologies in the class $\mathfrak{g}$.

Note the important distinction between *points on the graph* $\mathfrak{g}(\tau)$ as a (singular) manifold which will be denoted by small Latin letters $s, t, v \in \mathfrak{g}(\tau)$ and continuous "moduli" $\tau, T \in \mathbb{R}_+$, denoting *distances or lengths* on the graph. The location of the operator insertions is determined by the moduli. By choosing a coordinate system (parametrisation) of the graph resp. the propagators, a point $t \in \mathfrak{g}(\tau)$ can sometimes be assigned a number - its coordinate. The moduli, on the other hand, are independent of a choice of coordinates.

Instead of developing step-by-step the world graph formalism, we will state at once the respective form of the world graph path integral and prove subsequently that the amplitude obtained in this way is indeed identical to direct computation by the usual Feynman rules. Without loss of generality, we will study only connected graphs in the sequel. We need some technical tools to begin with.

Introduce for vector-valued functions $f, g : \mathfrak{g}(\tau) \to \mathbb{R}^d$ a real scalar product

$$\langle f, g \rangle \equiv \int_{\mathfrak{g}(\tau)} \mathrm{d}t \, f(t) \cdot g(t). \tag{3.1}$$

It is similarly defined for scalars. This product defines a real Hilbert space of functions on the metric graph $\mathfrak{g}(\tau)$.

The *graph Laplacian* is an operator acting on functions defined on the graph as a one-dimensional manifold with branching points at the vertices.[2] This is not the discrete graph Laplacian; we define the Laplacian $\triangle = \partial_t^2$ for functions on the graph as the one-dimensional continuous Laplacian along the parametrised links of the graph; at the vertices, we get a distributional contribution

$$\triangle f(t) = \sum_{\text{vertices } v} \left(\sum_{\text{adjacent links } l} \lim_{(s \text{ on } l) \to v} f'(s) \right) \delta_v(t) + \text{propagator contribs.} \tag{3.2}$$

(the Dirac distribution $\delta_v(t)$ on the graph is defined as

$$\int_{\mathfrak{g}(\tau)} \mathrm{d}t \, \delta_v(t) \, g(t) = g(v)$$

for a continuous function $g : \mathfrak{g}(\tau) \to \mathbb{R}^d$). While the first derivative of a function on the graph demands an orientation of the links, the second derivative is well-defined without this concept. The rule (3.2) applies also at vertices with only *one* internal propagator attached (such vertices have two or more external propagators attached).

Usually, we need a domain which makes the graph Laplacian a self-adjoint operator. The treatment of the graph Laplacian is not much different from the well-known treatment

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[2]These graphs have recently been termed "quantum graphs" in the physics community. For an introduction and overview, see [12] and references therein.

of the one-dimensional Laplacian $\triangle$ on the unit circle S^1, since the graphs we are considering are, with exception of the vertices, compact one-dimensional manifolds. A self-adjoint domain $\mathcal{D}(\triangle)$ can be constructed by closing the subspace of continuous functions with respect to the finite Sobolev norm $\|f\|_{H^2}^2 = \int_{\mathfrak{g}(\tau)} \mathrm{d}t \, [f^2 + (\partial_t f)^2 + (\triangle f)^2]$; on this domain, the graph Laplacian is symmetric by integration by parts (the marked difference to the S^1-case is the use of the graph derivative ∂_t in this Sobolev norm). Since the domain is maximal, this is also the domain of self-adjointness.

Let $t_j \in \mathfrak{g}(\tau)$ be the point on the graph where the external momentum k_j enters the graph; then in the world graph path integral, we have to include a factor $e^{-ik_j \cdot x(t_j)}$. For a concise notation, the external momentum "density" can be modelled by a generalised function

$$k\{\tau\} : t \mapsto \sum_j k_j \delta_{t_j}(t). \tag{3.3}$$

The argument $\{\tau\}$ indicates that the positions of the operator insertions are parametrised by the moduli. Thus we have

$$\sum_j k_j \cdot x(t_j) = \int_{\mathfrak{g}(\tau)} \mathrm{d}t \, x(t) \cdot k_j \delta_{t_j}(t) = \langle x, k\{\tau\} \rangle. \tag{3.4}$$

The contribution of all operator insertions in the path integral is then given by a factor $e^{-i\langle x, k\{\tau\}\rangle}$. We will start with the simpler situation where all propagators have the same mass $m(t) \equiv m$; then the mass term will contribute a factor $\exp\left(-m^2 |\mathfrak{g}(\tau)|\right)$, where $|\mathfrak{g}(\tau)|$ is the total length of the graph.

Theorem 1. *Let $\mathscr{G}$ be a compact Feynman graph. Let k_j, $j = 1 \ldots n$ be a collection of external momenta. The amplitude corresponding to the sum of all graphs in the equivalence class $\mathfrak{g} = [\mathscr{G}]$ is given by the formal world graph path integral*

$$G_{[\mathscr{G}]}(k_1, \ldots, k_n) = (2\pi)^{\frac{nd}{2}} \Big(\prod_{\text{vertices } v} -c_v \Big) \int_{\mathrm{Mod}\,\mathfrak{g}} \frac{\mathrm{d}^{\dim \mathfrak{g}} \tau}{\mathrm{Sym}(\mathfrak{g}(\tau))}$$
$$e^{-m^2 |\mathfrak{g}(\tau)|} Z_0(\mathfrak{g}(\tau))^{-1} \int_{C(\mathfrak{g}(\tau))} \mathscr{D}(x) \exp - \left(-\frac{1}{4} \langle x, \triangle x \rangle + i\langle x, k\{\tau\} \rangle \right), \quad (3.5)$$

where $Z_0(\mathfrak{g}(\tau))$ is a (formal) normalisation depending on the moduli.

The domain $C(\mathfrak{g}(\tau))$ is a reminder that the paths are supposed to be continuous at the vertices. We stress once more that the positions t_j of the operator insertions are part of the moduli. The normalisation $Z_0(\mathfrak{g}(\tau))$ will be determined below.

There is a subtlety concerning the symmetry factors in this formula: When the symmetry factors are taken over from the usual perturbative expansion, they may vary in general as the positions of the external insertions are varied. On the other hand, the symmetry factor could be determined with *all external insertions removed*. Since the symmetry factor is the size of the automorphism group of the graph, we would expect it to increase generally (since without the external insertions, there are less distinguishable features on the graph). When the external insertions are now again included, we have to take into account
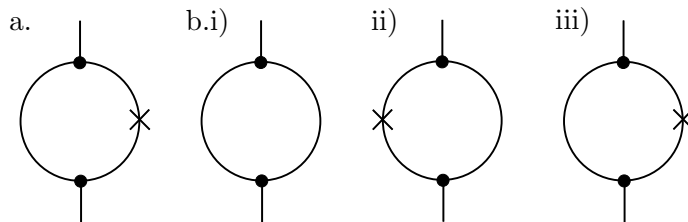
Figure 4: Example for symmetry factor counting. a. Subgraph from a bigger graph. The symmetry factor of this loop is 1, since there is an external insertion (X) which allows to distinguish both loop handles. b.i) The subgraph without the external insertion is divided by a symmetry factor 2, since both loop handles are indistinguishable. ii) + iii) The modulus determining the position of the external insertion may now place it on either handle of the loop. Since the resulting graphs are topologically equivalent, they add identically and just cancel the factor 1/2.

all possible insertion positions, and parametrise them by additional moduli. The point is that features of the graph which are *indistinguishable* from the point of view of the graph automorphism group are very well distinguishable from the point of view of the moduli. This causes an extra multiplicity which exactly cancels the surplus symmetry factors of the underlying graph without external insertions (see figure 4 for an illustration).

4. Interpretation as an effective theory of point particles carrying vector charges on the graph

In order to prove the theorem, we will reformulate the world graph path integral in such a manner that it will be seen to be equivalent to the partition function of a classical system of charged particles moving on the graph. This result is closely related to the results of Schmidt and Schubert [18], although we choose a different language (see also [13, 6]). It is reminiscent of the Born-Oppenheimer approximation to the hydrogen molecule, where after determining the effective potential for the nuclei mediated by the electrons, one analyses the motion of the nuclei in this effective potential (integration over moduli space). This is an interesting result in itself, but it will also aid the proof.

Consider the Gaussian integral in (3.5). The integration of the global translation degree of freedom yields the usual factor $(2\pi)^d \delta^{(d)}(\sum_j k_j)$, so

$$\int_{C(\mathfrak{g}(\tau))} \mathscr{D}(x) \exp - \left(-\frac{1}{4}\langle x, \triangle x \rangle + i\langle x, k\{\tau\} \rangle \right)$$
$$\sim (2\pi)^d \delta^{(d)}\left(\sum_j k_j \right) \exp \left\langle k\{\tau\}, \triangle^{-1} k\{\tau\} \right\rangle.$$

The exponent will be very important later on. It defines an "interaction potential" $V_{\text{eff}}(t_1, \ldots, t_n)$ of the operator insertions by

$$V_{\text{eff}}(t_1, \ldots, t_n) = - \left\langle k\{\tau\}, \triangle^{-1} k\{\tau\} \right\rangle \qquad \left(\sum_j k_j = 0 \right).$$

If we collect all normalisations into

$$Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) = Z_0(\mathfrak{g}(\tau))^{-1} \left\| \frac{-\triangle}{4\pi} \right\|_+^{-d/2} \tag{4.1}$$

(read further), then

$$Z_0(\mathfrak{g}(\tau))^{-1} \int_{C(\mathfrak{g}(\tau))} \mathscr{D}(x) \exp - \left(-\frac{1}{4}\langle x, \triangle x \rangle + i\langle x, k\{\tau\}\rangle \right)$$
$$= (2\pi)^d \delta^{(d)}\Big(\sum_j k_j \Big) Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) \, e^{-V_{\text{eff}}(t_1,\dots,t_n)}.$$

This formula deserves a few comments on the graph Laplacian $\triangle$. It is easy to see that $\triangle 1 = 0$, so the kernel of $\triangle$ is nonempty and $\triangle^{-1}$ is not uniquely defined in the first place. We therefore declare that we wish to study the $\triangle^{-1}$ obeying

$$\langle 1, \triangle^{-1} f \rangle = 0, \qquad f \in \text{Dom}(\triangle^{-1}) \tag{4.2}$$

(this scalar product is vector-valued, as the left hand side is a scalar and the right hand side is a vector). $-\triangle$ is a positive operator and its kernel consists of the constant functions. It has a pure point spectrum, as $\mathfrak{g}(\tau)$ is compact. In the determinant (4.1), we should therefore ignore the 0 eigenvalue on the right-hand side as it has been already taken care of in the explicit inclusion of momentum conservation. This is indicated by the symbol $\|.\|_+$.

On the other hand, $\langle 1, \triangle g \rangle = 0$; so $\triangle^{-1} f$ is only defined for f with $\langle 1, f \rangle = 0$. But

$$\Big\langle 1, k\{\tau\} \Big\rangle = \sum_j k_j = 0$$

precisely due to momentum conservation.

Naïvely, we would interpret the effective pair potential between two insertions $k_i \delta_{t_i}$ and $k_j \delta_{t_j}$ as

$$-2\langle k_i \delta_{t_i}, \triangle^{-1} k_j \delta_{t_j} \rangle,$$

however, as it stands, $\delta_{t_j} \notin \text{Dom}(\triangle^{-1})$ because $\langle 1, \delta_{t_j} \rangle = 1$. There is a canonical solution to the problem.

While δ_{t_i} is not in the domain of $\triangle^{-1}$, the difference $\delta_{ij} \equiv \delta_{t_i} - \delta_{t_j}$ certainly is. We may take advantage of this by employing repeatedly the momentum conservation condition $\sum_j k_j = 0$ (in the first and third equality) as

$$\Big\langle \sum_i k_i \delta_{t_i}, \; \triangle^{-1} \sum_j k_j \delta_{t_j} \Big\rangle = \Big\langle \sum_i k_i \delta_{i1}, \; \triangle^{-1} \sum_j k_j \delta_{j1} \Big\rangle = \sum_{i,j} (k_i \cdot k_j) \Big\langle \delta_{i1}, \; \triangle^{-1} \delta_{j1} \Big\rangle$$
$$= \frac{1}{2} \sum_{i,j} (k_i \cdot k_j) \Big\langle \delta_{i1} - \delta_{j1}, \; \triangle^{-1}(\delta_{j1} - \delta_{i1}) \Big\rangle = -\sum_{i<j} (k_i \cdot k_j)\langle \delta_{ij}, \; \triangle^{-1} \delta_{ij} \rangle.$$

Defining the pair potential

$$\varphi(t, t') = \langle \delta_t - \delta_{t'}, \; \triangle^{-1}(\delta_t - \delta_{t'}) \rangle, \tag{4.3}$$

the total effective potential is

$$V_{\text{eff}}(t_1, \ldots, t_n) = \sum_{i<j} (k_i \cdot k_j) \varphi(t_i, t_j). \tag{4.4}$$

While we have used total momentum conservation, we find a pair potential which is nevertheless independent of the positions of the other charges on the graph. It is a continuous function on $\mathfrak{g}(\tau) \times \mathfrak{g}(\tau)$ and bounded (so it is weak). By definition, $\varphi(t,t) = 0$. As $-\triangle^{-1}$ is a positive operator, in general $\varphi(t,t') \leq 0$. This implies that the interaction between parallel vector charges is a repulsive one, since $(k \cdot k)\varphi(t,t) \geq (k \cdot k)\varphi(t,t')$ for all positions $t, t' \in \mathfrak{g}(\tau)$ and charges $k \in \mathbb{R}^d$: Spatial separation of the charges is energetically favoured.

Putting everything together, the total amplitude (3.5) resulting from the equivalence class $\mathfrak{g} = [\mathscr{G}]$ can be written as

$$G_{[\mathscr{G}]}(k_1, \ldots, k_n) = (2\pi)^{\frac{nd}{2}} (2\pi)^d \delta^{(d)} \Big(\sum_j k_j \Big) \Big(\prod_{\text{vertices } v} -c_v \Big)$$

$$\int_{\text{Mod}\,\mathfrak{g}} \frac{Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau))\,\mathrm{d}^{\dim \mathfrak{g}}\tau}{\text{Sym}(\mathfrak{g}(\tau))} \exp\left(- \sum_{\text{vertices } i\,<\,j} (k_i \cdot k_j)\varphi(t_i, t_j) - m^2 |\mathfrak{g}(\tau)| \right) \tag{4.5}$$

(note that the interaction has been written here as a true pair potential, i.e. the sum extends over each unordered pair $\{i,j\}$ only once). The (vector valued) potential generated by all charges on the graph is

$$U_{\text{tot}}(t) = \sum_{\text{vertices } j} k_j \varphi(t, t_j), \qquad\qquad t \in \mathfrak{g}(\tau), \tag{4.6}$$

and we have

$$\sum_{\text{vertices } i\,<\,j} (k_i \cdot k_j)\varphi(t_i, t_j) = \frac{1}{2} \sum_{\text{vertices } i} k_i \cdot U_{\text{tot}}(t_i). \tag{4.7}$$

The potential on the graph fulfills

$$\triangle U_{\text{tot}}(t) = -2k\{\tau\}, \tag{4.8}$$

so in between the insertions, $U_{\text{tot}}(t)$ is a linear function (with vanishing second derivative).

4.1 Proof of theorem 1

The proof starts from the direct Schwinger representation (2.7) valid also for massless propagators. We introduce a matrix notation for the exponent. Define the symmetric covariance matrix $C \in M_V(\mathbb{R})$ as follows: if $v \neq w$, then

$$C_{vw} = -\frac{1}{2} \sum_{j(v \leftrightarrow w)} \tau_j^{-1} \qquad (v \neq w). \tag{4.9}$$

The sum extends over all propagators j connecting v and w directly; if there is no propagator connecting v and w directly then the matrix element $C_{vw} = 0$. The diagonal elements

are then chosen in such a way that the sum of each row/column equals zero. The matrix C is a linear combination of elementary matrices of the form

$$\mathbb{E}^{vw}_{ij} = \delta^v_i \delta^v_j + \delta^w_i \delta^w_j - \delta^v_i \delta^w_j - \delta^w_i \delta^v_j.$$

These matrices generate exactly the squares of the coordinate differences

$$x^T \mathbb{E}^{vw} x = (x_v - x_w)^2,$$

with the silent understanding that the entries $x_v \in \mathbb{R}^d$ of the vector x are themselves coordinate vectors. In terms of these building blocks,

$$C = \frac{1}{2} \sum_{v<w} \Big(\sum_{j(v \leftrightarrow w)} \tau_j^{-1} \Big) \mathbb{E}^{vw}. \tag{4.10}$$

Using the matrix C, we may write the amplitude (2.7) as

$$G_{\mathscr{G}}(k_1, \ldots, k_n) = \frac{(2\pi)^{\frac{nd}{2}}}{\mathrm{Sym}(\mathscr{G})} \Big(\prod_{\text{vertices } v} -c_v \int \mathrm{d}^d x_v \Big)$$
$$\Big(\prod_{\text{propagators } j} \int_0^\infty \frac{\mathrm{d}\tau_j}{(4\pi\tau_j)^{d/2}} \Big) \exp \Big(-\frac{1}{2} x^T C x - i k^T x - m^2 \sum_j \tau_j \Big).$$

We can say a few things about the spectrum of C. Let $e = \frac{1}{\sqrt{V}}(1, 1, \ldots, 1)^T$. By construction, $Ce = 0$. Because we consider connected graphs, the kernel of C contains only multiples of e. Furthermore, because $x^T C x$ is a sum of squares and never vanishes except when all x_v coincide, C is strictly positive with the exception of the eigenspace generated by e.

For the purpose of integrating $\mathrm{d}^d x_v$, we have to invert the singular matrix C. One can easily see that the 0 eigenvalue again enforces momentum conservation. The generalised inverse

$$C^{\mathrm{inv}} = \lim_{c \to \infty} (C + c e e^T)^{-1}.$$

always exists, because the eigenvector e decouples, and C is strictly positive elsewhere. When we insert the momentum-conserving δ-distribution in the end, we have to include a factor $V^{d/2}$ because of the normalisation of the eigenvector e. The amplitude is thus

$$G_{\mathscr{G}}(k_1, \ldots, k_n) = \frac{(2\pi)^{\frac{nd}{2}}}{\mathrm{Sym}(\mathscr{G})} (2\pi)^d \delta^{(d)} \left(\sum_j k_j \right) \left(\prod_{\text{vertices } v} -c_v \right)$$
$$\left(\prod_{\text{propagators } j} \int_0^\infty \frac{\mathrm{d}\tau_j}{(4\pi\tau_j)^{d/2}} \right) V^{d/2} \left\| \frac{C}{2\pi} \right\|_+^{-d/2} \exp \left(-\frac{1}{2} k^T C^{\mathrm{inv}} k - m^2 \sum_j \tau_j \right). \tag{4.11}$$

The non-singular part of the determinant can be obtained by

$$\|C\|_+ = \det(C + e e^T).$$

Noting that the "ordinary" Schwinger parameters τ_j which we have used are in one-to-one correspondence to the continuous moduli of the equivalence class scheme B, we can easily obtain a sum over the classes of scheme A or scheme C by summing over the necessary orderings of the operator insertions, and over the discrete moduli defining the permutations of the insertions, and possibly over graph topologies (this is possible since the equivalence classes B are subclasses of A and C). Hence, equality with theorem 1 is established if we can show that the exponential coincides with the one in (4.5), i.e. if

$$C_{ij}^{\text{inv}} = \varphi(t_i, t_j). \tag{4.12}$$

C^{inv} should be the matrix analog to the pair potential φ. Rather than proving this formula directly, we compute

$$\sum_{jl} C_{ij} \varphi(t_j, t_l) k_l = \sum_j C_{ij} U_{\text{tot}}(t_j)$$

$$= \frac{1}{2} \sum_{\substack{\text{propagators } l \\ \text{ending at vertex } 2(l) = i}} \frac{U_{\text{tot}}(t_i) - U_{\text{tot}}(t_{1(l)})}{\tau_l} = \frac{1}{2} \sum_{\substack{\text{prop.s } l \\ \text{with } 2(l) = i}} U_{\text{tot}}'(t)\Big|_{t \text{ on } l} = k_i$$

by the fact that the potential is linear along the propagators, and equation (4.8). This proves (4.12). By comparison, we find that the normalisation constant must be given by

$$Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) = V^{d/2} \left\| \frac{C}{2\pi} \right\|_+^{-d/2} \prod_{\text{propagators } j} \frac{1}{(4\pi\tau_j)^{d/2}}. \tag{4.13}$$

This concludes the proof.

As a side-effect, we have found a closed formula for the normalisation $Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau))$. In many cases, the following lemma states a more useful form:

Lemma 2. *The measure $Z_{eff}^{-1}(\mathfrak{g}(\tau)) \, \mathrm{d}^{\dim \mathfrak{g}}\tau$ on moduli space $\mathrm{Mod}\,\mathfrak{g}$ is given by*

$$Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) = \Big(\prod_{\text{vertices } v} \int \mathrm{d}^d x_v \Big) \delta^{(d)}(x_1) \prod_{\text{propagators } j} \left(\frac{1}{4\pi\tau_j} \right)^{d/2} e^{-\frac{1}{4\tau_j}(x_{1(j)} - x_{2(j)})^2},$$

where $x_{1(j)}$ and $x_{2(j)}$ are the endpoints of the propagators and x_1 is an arbitrary vertex on the graph.

Proof. Equate (4.11) and (2.7). Pick an arbitrary vertex x_1. Integrate $\int \mathrm{d}^d k_1$ and put all other external momenta k_j to zero, $j = 2 \ldots V$. The proposed expression is obtained by substituting equation (4.13). $\qquad\square$

4.2 Coincidence of vertices: cell structure of moduli space. Renormalisation

If the modulus τ_j associated to any one propagator in a graph $\mathfrak{g}(\tau)$ shrinks to 0 (see figure 5), the two adjacent vertices concur in the limit. It is important to realise that there is a fundamental difference between the limit $\tau_j \to 0$ of the graph $\mathfrak{g}(\tau)$, and the graph $\mathfrak{g}'(\tau')$ containing a single vertex, obtained through fusion of the pair of adjacent vertices (τ' are
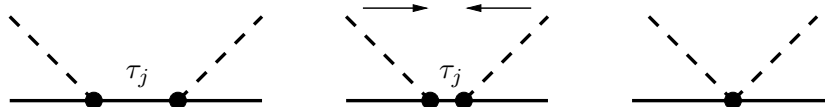
Figure 5: (left) Section of metric graph $\mathfrak{g}(\tau)$ containing a propagator with modulus τ_j. The amplitude contains the product c_3^2 of coupling constants. (middle) While $\tau_j \to 0$, the insertions are approaching each other. Still, the amplitude is proportional c_3^2. (right) After the insertions fuse, the new metric graph $\mathfrak{g}'(\tau')$ is proportional c_4.

"reduced" moduli, removing τ_j from the moduli τ). While the modulus τ_j is a continuous parameter, a coordinate parametrising the integrand of the moduli space integration, the hypersurface $\tau_j = 0$ lies on the boundary of $\mathrm{Mod}\,\mathfrak{g}$ and therefore has measure zero; so it does not contribute (for divergences, so below).

In contrast, the class $\mathfrak{g}'$ - viewed as an independent contribution to the total correlation - has a nonvanishing measure in general. That $\mathfrak{g}'(\tau')$ and $\mathfrak{g}(\tau)|_{\tau_j=0}$ are truly different can also be seen from that fact that the valency of the fused vertex and therefore the prefactor assembled from the product of coupling constants is different. In technical terms, $\mathfrak{g}'$ specifies a different cell of moduli space.

It is interesting to study the global structure of the complete moduli space, and examine the relations between different cells. Following the literature, we claim that $\mathrm{Mod}\,\mathfrak{g}' \subset \partial\,\mathrm{Mod}\,\mathfrak{g}$, i.e. the cell resulting if one (or several) moduli $\tau_j \to 0$ makes up part of the boundary of the original cell [10]. For the dimensions of the adjacent cells $\mathfrak{g}$ and $\mathfrak{g}'$, obviously $\dim \mathfrak{g}' < \dim \mathfrak{g}$. In this way, the moduli space has a natural complex structure (in the topological sense).

Another kind of boundary is reached in the limit $\tau_j \to \infty$: Momentum transfer through the propagator is increasingly suppressed; this cell boundary is made up of the graph where the propagator is missing out altogether. Note that the limiting graph still contains *two* propagators on both sides, as well as additional 2-valent vertices (mass terms). It may happen that the graph falls apart into two components in this limit; a systematic treatment therefore has to include disconnected graphs.

This opens the door for speculations whether there occurs an ultimate simplification in the amplitude when we extend the sum over all different cells of moduli space (prior to integration of the moduli). There is one context where this is indeed required, namely in renormalisation. Rewriting the naïve Schwinger parametrisation of the propagator as

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^\infty \mathrm{d}\tau_j \, e^{-\tau_j(q^2+m^2)} = \lim_{\varepsilon \to 0+} \frac{e^{-\varepsilon(q^2+m^2)}}{q^2 + m^2}, \qquad (4.14)$$

in the limit $\varepsilon \to 0$, the suppression of high q^2 momentum contributions due to the regularising exponent vanishes and a UV divergence is a possible consequence. We make the following

Assumption. Cancellations due to renormalisation are *local* in moduli space.

It is immediate that the necessary (formally infinite) counterterms must come from the neighbouring moduli space cell reached in the limit $\tau_j \to 0$.

For IR divergences, rewrite the Schwinger parametrisation of the massless propagator as

$$\lim_{\varepsilon \to \infty} \int_0^\varepsilon \mathrm{d}\tau_j \, e^{-\tau_j q^2} = \lim_{\varepsilon \to \infty} \frac{1 - e^{-\varepsilon q^2}}{q^2}; \qquad (4.15)$$

it is the limit $\tau_j \to \infty$ which is responsible for a possible IR divergence at $q^2 \to 0$. Correspondingly, the IR "counterterms" are to be obtained from the diagram without the propagator in question. It has a regularisation mass term coming from the unobservable background modes coupling to the fields.

Viewing renormalisation in the geometrical moduli space picture, it is obvious that the cancellation of divergences is independent of the particular moduli space parametrisation. A renormalisation example will be given in section 4.4.

As an intriguing possibility, it is imaginable that the moduli space integrations can be formulated as integrals of a total divergence; in that case, by Stokes's Theorem, they might be reduced to an integral over boundary terms *only*, even if we cannot expect the boundary terms to have a direct graphical interpretation. By iteration of this procedure, amplitudes could be computed as integrals over the lowest dimensional boundary cells of moduli space (moduli space "effective vertices").[3]

4.3 Example: tree diagrams

The most elementary examples of connected Feynman graphs are tree diagrams. The vertices (insertions) at the "endpoints" of the tree are connected to the rest of the diagram only by a single internal line (so they have internal valency 1), and they are linked to at least two external lines (external valency ≥ 2). "Internal" branching vertices (internal valency ≥ 3) may or may not have a non-zero external valency as well. Finally, there are vertices with internal valency two and external valency ≥ 1.

To cover the whole equivalence class $\mathfrak{g}$ of the tree, we have to sum over all topologically inequivalent groupwise permutations of the external legs whenever several external legs are located on insertions with the same valences; insertions with internal valency 2 are allowed to slide all over the tree. As the external legs are all distinguishable, there are no symmetries of the graph, so the symmetry factor equals 1.

Let there be n insertions with internal valency 2. We denote their positions on the metric graph by $t_j \in \mathfrak{g}(\tau)$, and the total external momentum entering at insertion j by k_j. Removing these, let there be V vertices left with internal valency other than 2, connected by P lines. Denote the lengths of these lines by the moduli $\tau_j \in \mathbb{R}_+$. Clearly, the dimension of the graph (the number of continuous moduli) is $\dim \mathfrak{g} = n + P$.

Given any two insertions $t_1, t_2 \in \mathfrak{g}(\tau)$, we want to find the interaction potential for the momenta entering at these insertions (the charges). We have to determine

$$f(t) = \triangle^{-1}(\delta_{t_1} - \delta_{t_2})(t), \qquad t \in \mathfrak{g}(\tau).$$

---

[3]Such iteration might require very particular relations between the coupling constants and masses of the system, as in non-Abelian gauge theories. The case of gauge theories is indeed special: Here, the couplings are fixed ab initio by the requirement of gauge invariance.

The function $f(t)$ is easily characterized: Let $\mathcal{T}(t_1, t_2)$ denote the path from t_1 to t_2, and $|\mathcal{T}(t_1, t_2)|$ be its length (in terms of the moduli τ_j). $f(t)$ is continuous by definition; it is piecewise constant on all segments of the graph except $\mathcal{T}(t_1, t_2)$; and on $\mathcal{T}(t_1, t_2)$ it increases linearly with the distance from t_1. The absolute value of $f(t)$ is unimportant, so an arbitrary constant may be added. It follows that

$$\varphi(t_1, t_2) = \langle \delta_{t_1} - \delta_{t_2}, \triangle^{-1}(\delta_{t_1} - \delta_{t_2}) \rangle = f(t_1) - f(t_2) = -|\mathcal{T}(t_1, t_2)|.$$

For the effective normalisation, one can see that $Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) = 1$ by starting to integrate the formula given in lemma 2 at the vertices forming the tips of the tree, and working down towards v_1 which is an arbitrary vertex on the tree. The integrals then always cancel exactly the prefactor.

So for a tree diagram, we find the amplitude

$$G_{\text{tree}}(k_1, \ldots, k_n) = (2\pi)^{\frac{nd}{2}}(2\pi)^d \delta^{(d)}\left(\sum_j k_j\right)\left(\prod_{\text{vertices } v} -c_v\right)$$

$$\sum_{\text{perm.s of } k_j} \int_0^\infty \mathrm{d}^P\tau \int_{\mathfrak{g}(\tau)} \mathrm{d}^n t \, \exp\left(\sum_{v<w}(k_v \cdot k_w)|\mathcal{T}(t_v, t_w)| - m^2 \sum_j \tau_j\right).$$

It is an amusing exercise to see how this expression resolves into the correct amplitude upon integrating the moduli.

4.4 Example: the one-loop "spider" diagram

The simplest diagram containing a loop integration is the one-loop amplitude with an arbitrary number n of external legs[4] coupled by three-valent vertices c_3 directly to the loop. The class $\mathfrak{g}$ contains all permutations of the order of external legs; there is a cyclic symmetry.

Let the incoming momenta be $k_1, \ldots, k_n$. Let t_j be the coordinate which describes the position of the j-th vertex entering the loop with respect to some fixed parametrisation of the loop. Furthermore, let τ be the total length of the loop. The true moduli are given by the distances *between* the insertions; so if we integrate the coordinates t_j instead, we have to include a factor $\frac{1}{\tau}$ to account for the arbitrary choice of an origin of the parametrisation. For the normalisation, one finds

$$Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) = \left(\frac{1}{4\pi\tau}\right)^{d/2}$$

by convolution of the Wiener kernels of lemma 2. Assume that $t_i < t_j$. Then,

$$\triangle^{-1}(\delta_{t_i} - \delta_{t_j})(t) = \begin{cases} \left(\frac{t_j - \tau + t_i}{2} - t\right)\frac{t_j - t_i}{\tau} & \text{if } t < t_i, \\ \left(t - \frac{t_i + t_j}{2}\right)\frac{\tau - t_j + t_i}{\tau} & \text{if } t_i \leq t < t_j, \\ \left(\frac{t_j + \tau + t_i}{2} - t\right)\frac{t_j - t_i}{\tau} & \text{if } t_j \leq t \end{cases}$$

---

[4]A yet unknown species of spiders.

at the point with coordinate t with respect to the parametrisation. This is easily checked by applying $\triangle$. Thus,

$$\varphi(t_i, t_j) = \langle \delta_{t_i} - \delta_{t_j}, \triangle^{-1}(\delta_{t_i} - \delta_{t_j}) \rangle = -\frac{|t_j - t_i|(\tau - |t_j - t_i|)}{\tau}.$$

In the last form, the potential is valid also for $t_i > t_j$. As the orientation of the loop parametrisation is arbitrary, we have to include a symmetry factor $\frac{1}{2}$, and we get a total amplitude

$$G_{\text{1-loop}}(k_1, \ldots, k_n) = (2\pi)^{\frac{nd}{2}} (2\pi)^d \delta^{(d)}\Big(\sum_j k_j \Big) (-c_3)^n \int_0^\infty \frac{\mathrm{d}\tau}{2\tau} \left(\frac{1}{4\pi\tau} \right)^{d/2}$$

$$\exp\left(-m^2\tau\right) \int_0^\tau \mathrm{d}^n t \, \exp\left(\sum_{i<j}(k_i \cdot k_j)\frac{|t_j - t_i|(\tau - |t_j - t_i|)}{\tau} \right). \quad (4.16)$$

Written in this form, it is plausible that the presummation of sufficiently large equivalence classes of diagrams amounts to a stringent organisation of the amplitudes. Had we used the equivalence class scheme B (the world line formalism), the cyclic order of external legs entering the loop would be unalterable, and we should sum explicitly over all such orderings.

This provides a simple example illustrating how renormalisation fits into the scheme: By rescaling the parameters $t_j \to \tau t_j$, the integral takes the form

$$G_{\text{1-loop}}(k_1, \ldots, k_n) = (2\pi)^{\frac{nd}{2}} (2\pi)^d \delta^{(d)}\Big(\sum_j k_j \Big) (-c_3)^n \int_0^\infty \frac{\mathrm{d}\tau}{2\tau} \left(\frac{1}{4\pi\tau} \right)^{d/2} \tau^n$$

$$\exp\left(-m^2\tau\right) \int_0^1 \mathrm{d}^n t \, \exp\left(\tau \sum_{i<j}(k_i \cdot k_j)|t_j - t_i|(1 - |t_j - t_i|) \right).$$

The integrand has a power series expansion in τ around the origin

$$c_{n-d/2-1}\tau^{n-d/2-1} + c_{n-d/2}\tau^{n-d/2} + c_{n-d/2+1}\tau^{n-d/2+1} + \ldots; \quad (4.17)$$

the integral diverges at $\tau \to 0$ if $n - d/2 \leq 0$. This is a UV divergence, by the reasoning of section 4.2. The graphs containing the counterterms are found in the moduli space cell $\tau \to 0$. They are just given by formally integrating the divergent terms of the power series expansion (4.17):

$$G_{n-d/2-1}(k_1, \ldots, k_n) = -(2\pi)^{\frac{nd}{2}} (2\pi)^d \delta^{(d)}\Big(\sum_j k_j \Big) (-c_3)^n \int_0^\infty \frac{\mathrm{d}\tau}{2\tau} \left(\frac{1}{4\pi\tau} \right)^{d/2} \tau^n,$$

$$G_{n-d/2}(k_1, \ldots, k_n) = -(2\pi)^{\frac{nd}{2}} (2\pi)^d \delta^{(d)}\Big(\sum_j k_j \Big) (-c_3)^n \int_0^\infty \frac{\mathrm{d}\tau}{2\tau} \left(\frac{1}{4\pi\tau} \right)^{d/2} \tau^{n+1}$$

$$\left(-m^2 + (k_i \cdot k_j) \int_0^1 \mathrm{d}^n t \sum_{i<j} |t_j - t_i|(1 - |t_j - t_i|) \right),$$

etc. Naturally, the counterterms for higher order divergences are of derivative type. With these subtractions in place, the total amplitude is finite.
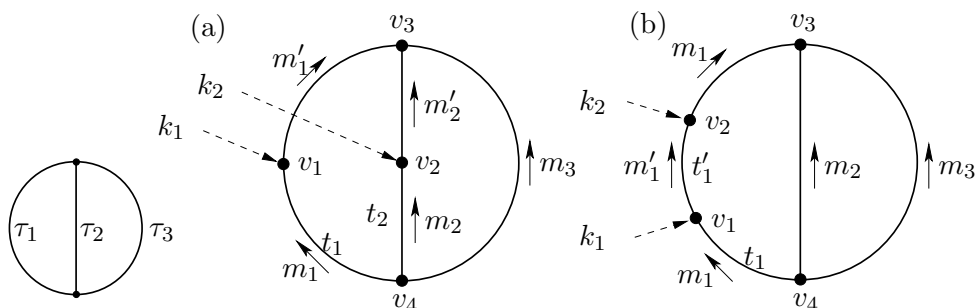
Figure 6: Notation in the two-loop computation. v_1, v_2: external insertions. k_1, k_2: external momenta. v_3, v_4: internal vertices. τ_1, τ_2, τ_3 (inset): length moduli of branches between v_1 and v_2. t_1, t_1', t_2: length moduli of segments between vertices / insertions. $m_1, m_2, \ldots$, arrows: gradients of the piecewise linear function $\triangle_v^{-1}(\delta_{v_1} - \delta_{v_2})(v)$ along the branches of the graph.

4.5 Example: two-loop self-energy graph

At two-loop level, the sliding of external legs over internal vertices is important. We discuss the contribution of the graph given in figure 6 to the self-energy in scalar ϕ^3-theory.

There are two different topologies: Both external insertions v_1 and v_2 may slide along the same internal line between the two vertices (a), or they may be along two separate lines (b). With the moduli parametrising the metric distances along the propagators as defined in the figure, the first task is the computation of the pair potentials $\varphi_{(a)}(v_1, v_2)$ resp. $\varphi_{(b)}(v_1, v_2)$, as defined in (4.3).

We need to compute the action of the inverse Laplacian $\triangle_v^{-1}(\delta_{v_1} - \delta_{v_2})(v)$ for points $v \in \mathfrak{g}(\tau)$ on the graph. We surely know that this is a continuous function, piecewise linear between vertices. Denote the respective gradients $\partial_v[\triangle_v^{-1}(\delta_{v_1} - \delta_{v_2})](v)$ in (a) by $m_1, m_1', m_2, m_2', m_3$, as in the figure. These gradients fulfill the equations

$$m_1' - m_1 = 1$$
$$m_2' - m_2 = -1$$
$$t_1 m_1 + (\tau_1 - t_1)m_1' = t_2 m_2 + (\tau_2 - t_2)m_2' = \tau_3 m_3$$
$$m_1 + m_2 + m_3 = 0$$

(the first pair of equations normalises the δ source terms; the second pair says that the potential is continuous at the vertices v_3 and v_4; and the last equation declares that v_4 is uncharged). There is a another similar but redundant condition at the vertex v_3. The solution is, with $\Delta = \tau_1 \tau_2 + \tau_1 \tau_3 + \tau_2 \tau_3$,

$$m_1' = \frac{(\tau_2 + \tau_3)t_1 + \tau_3 t_2}{\Delta} \qquad\qquad m_1 = m_1' - 1$$
$$m_2' = -\frac{(\tau_1 + \tau_3)t_2 + \tau_3 t_1}{\Delta} \qquad\qquad m_2 = m_2' + 1$$
$$m_3 = \frac{\tau_1 t_2 - \tau_2 t_1}{\Delta}$$

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For the pair potential between insertions v_1 and v_2 in case (a), this implies

$$\varphi_{(a)}(v_1, v_2) = \langle \delta_{v_1} - \delta_{v_2}, \triangle^{-1}(\delta_{v_1} - \delta_{v_2}) \rangle = t_1 m_1 - t_2 m_2 \qquad (4.18)$$
$$= -\frac{\tau_1(\tau_2 - t_2)t_2 + \tau_2(\tau_1 - t_1)t_1 + \tau_3(t_1 + t_2)(\tau_1 + \tau_2 - t_1 - t_2)}{\Delta}.$$

A similar computation for case (b) reveals that

$$\varphi_{(b)}(v_1, v_2) = -\frac{t_1'\left[(\tau_2 + \tau_3)(\tau_1 - t_1') + \tau_2\tau_3\right]}{\Delta}. \qquad (4.19)$$

This second potential does not depend on the modulus t_1.

These potentials are really only pieces or "branches" of a *single* potential function $\varphi(v_1, v_2)$, continuous for all $v_1 \times v_2 \in \mathfrak{g}(\tau) \times \mathfrak{g}(\tau)$. They can be connected e.g. when the external insertion v_2 crosses through v_3:

$$\lim_{v_2 \to v_3} \varphi_{(a)}(v_1, v_2) = \lim_{t_2 \to \tau_2} \varphi_{(a)}(v_1, v_2) = -\frac{(\tau_1 - t_1)\left[(\tau_2 + \tau_3)t_1 + \tau_2\tau_3\right]}{\Delta}$$
$$= \lim_{t_1' \to \tau_1 - t_1} \varphi_{(b)}(v_1, v_2) = \lim_{v_2 \to v_3} \varphi_{(b)}(v_1, v_2).$$

A second way to pass from sheet (a) to sheet (b) is to pass v_1 through v_4 and use the symmetry of the graph. Again, the potential is continuous.

The computation of the normalisation constant Z_{eff}^{-1} by equation (4.13) is straightforward. There are $V = 2$ vertices in the naked graph; the matrix C according to (4.10) is

$$C = \frac{1}{2}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_2}\right) \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix};$$

and thus with $\|C\|_+ = \det(C + ee^T) = 2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_2})$, the normalisation becomes

$$Z_{\text{eff}}^{-1}(\mathfrak{g}(\tau)) = \left(\frac{1}{32\pi^2\Delta}\right)^{d/2}.$$

The symmetry factors for both graphs are 2. The total amplitude is therefore

$$G_{\text{2-loop}}(k_1, k_2) = (2\pi)^{2d}\delta^{(d)}\left(k_1 + k_2\right) \frac{(-c_3)^4}{2} \int_0^\infty \mathrm{d}^3\tau \left(\frac{1}{32\pi^2(\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)}\right)^{d/2}$$
$$e^{-m^2(\tau_1 + \tau_2 + \tau_3)} \cdot \left\{ \int_0^{\tau_1} \mathrm{d}t_1 \int_0^{\tau_2} \mathrm{d}t_2 \, \exp\left(-(k_1 \cdot k_2)\varphi_{(a)}(v_1, v_2)\right) \right.$$
$$\left. + \int_0^{\tau_1} \mathrm{d}t_1' \, (\tau_1 - t_1') \exp\left(-(k_1 \cdot k_2)\varphi_{(b)}(v_1, v_2)\right) \right\}. \qquad (4.20)$$

The factor $\tau_1 - t_1' = \int_0^{\tau_1 - t_1'} \mathrm{d}t_1$ comes from integrating the irrelevant modulus t_1.

4.6 Particles of different mass

We comment briefly on how to include particles of different mass. For being explicit, consider the self-energy diagram figure 1. Assume that the internal propagators carry different masses m_1 and m_2, and that the vertices always couple to one external and one of each particles of masses m_1, m_2. The total mass exponential can then be written

$$
e^{-m_1^2 \tau_1 - m_2^2 \tau_2} + e^{-m_2^2 \tau_1 - m_1^2 \tau_2}
$$

$$
= \mathrm{Tr} \left\{ \begin{pmatrix} e^{-m_1^2 \tau_1} & 0 \\ 0 & e^{-m_2^2 \tau_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-m_1^2 \tau_2} & 0 \\ 0 & e^{-m_2^2 \tau_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.
$$

The first and third matrix are representing the propagators; the other two matrices are the vertices "switching" between different masses. This might seem overly formal. However, it is only consequent. Namely, observe that this is a trace over a path-ordered exponential

$$
\mathrm{Tr}\, P_{\mathfrak{g}(\tau)} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{t_1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{t_2} \exp - \int_0^\tau \mathrm{d}t \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \right\},
$$

where t_1 and t_2 denote the locations of the operator insertions, τ is the total length of the loop, and $P_{\mathfrak{g}(\tau)}$ is the path ordering on $\mathfrak{g}(\tau)$. This construction can be generalised to more complex graphs (although a matrix notation for the vertices is clearly not possible). The resulting mass exponential is then part of the integrand of the moduli space integral, in the spirit of the world graph formalism. Similar techniques can be applied if the propagators are gauge bosons in the adjoint representation of some local gauge group (for the application of path ordering to inclusion of background potentials see [7]).

5. Vector and tensor particles and the dipole method

So far, we have considered scalar interaction vertices and scalar particles. In general, vertices might contain derivatives of the adjoining propagators; when tensor particles are involved, their propagators will contain supplementary polynomials in their momenta, and the propagator term $(q_j^2 + m^2)^{-1}$ might be raised to a higher power. While the latter can be generated by including additional factors τ_j in the measure on moduli space (cf. section 2.2), there remains a prefactor multiplying the integrand of a Feynman amplitude which is a polynomial in the momenta q_i of the propagators and k_i of the external legs. We develop here a formalism which introduces a generating function for the prefactor, i.e. a function

$$
G(k_1, \ldots, k_n; y_1, \ldots, y_p)
$$

where each y_i is associated to one internal propagator of the graph. For $y_i \equiv 0$ the function G is equal to the usual correlation function without prefactor

$$
G(k_1, \ldots, k_n; 0, \ldots, 0) = G(k_1, \ldots, k_n) = \int \mathrm{d}^{\ell d} p^{\mathrm{loop}} F(k, p^{\mathrm{loop}});
$$

here p_i^{loop} are ℓ loop momentum variables, to be integrated over. Let $P(q_1, \ldots, q_p)$ be a polynomial of the internal momenta along the propagators (in general we will not assume
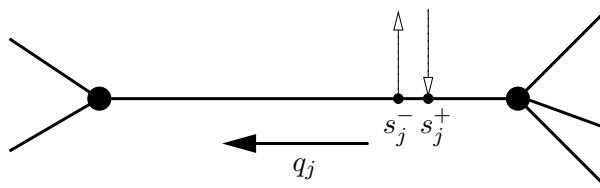
Figure 7: Insertion of a test dipole onto an internal propagator.

that vector indices are contracted completely). If we substitute the (components of the) y-derivatives for the internal momenta q in the polynomial P and apply it to the generating function

$$P\left(i\frac{\partial}{\partial y}\right)G(k;y)\bigg|_{y=0} = \int \mathrm{d}^{\ell d}p^{\mathrm{loop}}P(q)F(k,p^{\mathrm{loop}}),\qquad(5.1)$$

we should obtain the prefactor given by the polynomial $P(q)$. For tree graphs, the momenta q_i are constant; for loop graphs, there is an integration over loop momenta.

There exists a method in the "vector charge" framework to reproduce the corresponding amplitudes. It utilizes the insertion of an additional pair of oppositely charged external legs on each propagator - a dipole. We are given a Feynman graph $\mathfrak{g}(\tau)$ containing various propagators and vertices in the Schwinger parametrised form with the loop momenta not yet integrated. Pick a propagator j transporting momentum q_j from vertex v_1 to vertex v_2. On j, we want to insert a dipole with strength $\frac{iy_j}{2}$. The dipole is constructed from two infinitesimally separated sources: Source s_j^+ located arbitrarily within the propagator and source s_j^- at distance ε away from s_j^+ in the direction of q_j as we define it (see figure 7). As ε is supposed to be infinitesimal, s_j^- will always fit on the propagator. In the limit $\varepsilon \to 0$, the position of the dipole on the propagator becomes s_j. At s_j^+, momentum $\frac{iy_j}{2\varepsilon}$ flows into the graph; at s_j^-, momentum $\frac{iy_j}{2\varepsilon}$ flows out of the graph; so the total momentum is conserved. The "dipole moment" is the vector $\mu_j = \varepsilon \cdot \frac{iy_j}{2\varepsilon} = \frac{iy_j}{2}$. Writing down the local momentum balance, it is clear that there is a momentum transfer $q_j + \frac{iy_j}{2\varepsilon}$ between s_j^+ and s_j^-. In addition, we multiply the integrand by a "dipole self-energy renormalisation constant" $e^{-\frac{y_j^2}{4\varepsilon}}$. Without the dipole, the piece between the insertions naturally contributes $e^{-\varepsilon q_j^2}$ to the Schwinger parametrised path integral. The total effect of the dipole including the self-energy renormalisation is a correction factor

$$e^{\varepsilon q_j^2 - \varepsilon(q_j + \frac{iy_j}{2\varepsilon})^2 - \frac{y_j^2}{4\varepsilon}} = e^{-iy_j \cdot q_j}.$$

This shows that the amplitude is indeed a generating function for polynomials in q_j by (5.1).

We saw that after integration of the loop momenta, the exponent in the Schwinger integrand (4.5) is equivalent to the potential of an ensemble of vector charges defined by the external momenta. The dipole insertions fit quite naturally into this picture. Because they can be assembled from "elementary" vector charges, the available ingredients of the formalism are completely sufficient to handle them. All we have to do is calculate the additional potential terms which arise due to the insertion of the dipoles (for each propagator).

It is advantageous to compute the dipole-scalar and dipole-dipole pair potentials before summing over all pairs. We define

$$\varphi^{\mathrm{ds}}(s_j, v_i) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\varphi(s_j^+, v_i) - \varphi(s_j^-, v_i) \right) = -\frac{\partial}{\partial s_j} \varphi(s_j, v_i) \tag{5.2}$$

$$\varphi^{\mathrm{dd}}(s_j, s_i) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\varphi(s_j^+, s_i^+) - \varphi(s_j^-, s_i^+) - \varphi(s_j^+, s_i^-) + \varphi(s_j^-, s_i^-) \right)$$

$$= -\frac{\partial}{\partial s_i} \varphi^{\mathrm{ds}}(s_j, s_i) = \frac{\partial^2}{\partial s_j \partial s_i} \varphi(s_j, s_i). \tag{5.3}$$

where the last equation is valid only for $i \neq j$ (the limits $\varepsilon \to 0$ are allowed because it is easy to see that both potentials are of order $\mathcal{O}(\varepsilon^0)$ and the higher-order terms are not relevant). The dipole-scalar energy is $\mu_i \cdot k_j \varphi^{\mathrm{ds}}(s_j, v_i)$, and similarly for dipole-dipole. For the self-interaction of a dipole, we define

$$\varphi_\varepsilon^{\mathrm{d\text{-}self}}(s_j) = -\frac{1}{\varepsilon^2} \varphi(s_j^-, s_j^+). \tag{5.4}$$

The full generating function is then obtained by inserting the factor

$$\exp - \left[\frac{i}{2} \sum_{\mathrm{prop.s}\ j} \sum_{\mathrm{vert.s}\ i} (y_j \cdot k_i) \varphi^{\mathrm{ds}}(s_j, v_i) + \left(\frac{i}{2} \right)^2 \sum_{\mathrm{prop.s}\ i\ <\ j} (y_i \cdot y_j) \varphi^{\mathrm{dd}}(s_i, s_j) \right.$$

$$\left. + \left(\frac{i}{2} \right)^2 \sum_{\mathrm{prop.s}\ j} y_j^2 \left(\varphi_\varepsilon^{\mathrm{d\text{-}self}}(s_j) - \frac{1}{\varepsilon} \right) \right].$$

By construction, this must be independent of ε, and even of the coordinates s_j of the dipole insertions on the propagators.[5]

We apply this formula to the loop graph. We keep the nomenclature of section 4.4 introducing the loop graph and insert the dipoles at the general loop coordinates s_j. The dipoles are oriented along the canonical loop direction. We introduce the oriented distance function

$$[0, T[\ni \tau_{ji} = (s_j - v_i) \mod T.$$

The relevant potentials are then

$$\varphi(v_j, v_i) = -\frac{\tau_{ji}(T - \tau_{ji})}{T}, \tag{5.5}$$

$$\varphi^{\mathrm{ds}}(s_j, v_i) = -\frac{\partial}{\partial s_j} \frac{-\tau_{ji}(T - \tau_{ji})}{T} = \frac{T - 2\tau_{ji}}{T},$$

$$\varphi^{\mathrm{dd}}(s_j, s_i) = -\frac{\partial}{\partial s_i} \frac{T - 2\tau_{ji}}{T} = -\frac{2}{T},$$

$$\varphi_\varepsilon^{\mathrm{d\text{-}self}}(s_j) = \frac{1}{\varepsilon} \frac{T - \varepsilon}{T}.$$

The dipole self-energy is divergent as $\varepsilon \to 0$; however if we include the additional factor $e^{-\frac{y_j^2}{4\varepsilon}}$ in the action, then the "renormalised" self-energy

$$\varphi^{\mathrm{d\text{-}self(ren)}}(s_j) = \lim_{\varepsilon \to 0} \left(\varphi_\varepsilon^{\mathrm{d\text{-}self}}(s_j) - \frac{1}{\varepsilon} \right) = -\frac{1}{T}$$

---

[5]As the field is constant along the propagator.

has a reasonable limit for vanishing dipole extension. The complete generating factor is therefore

$$\exp\left\{-\frac{i}{2}\sum_{i,j=1}^{n}(k_i\cdot y_j)\frac{T-2\tau_{ji}}{T}-\frac{1}{4T}\Big(\sum_{j=1}^{n}y_j\Big)^2\right\}.$$

The generating function is therefore

$$G_{\text{1-loop}}(k_1,\ldots,k_n;y_1,\ldots,y_n)=(2\pi)^d\delta^{(d)}\Big(\sum_j k_j\Big)(-c_3)^n\int_0^\infty\frac{\mathrm{d}T}{2T}\left(\frac{1}{4\pi T}\right)^{d/2}$$

$$\exp\left(-m^2 T\right)\int_0^T\mathrm{d}^n t\,\exp\left\{\sum_{i<j}(k_i\cdot k_j)\frac{|t_j-t_i|(T-|t_j-t_i|)}{T}\right\}$$

$$\exp\left\{-\frac{i}{2}\sum_{i,j=1}^{n}(y_j\cdot k_i)\frac{T-2\tau_{ji}}{T}-\frac{1}{4T}\Big(\sum_{j=1}^{n}y_j\Big)^2\right\},$$

Due to the fact that we chose the test dipoles to be imaginary, the last factor is well-behaved for real y_j when integrating the modulus T. Note, however, that there is an essential singularity at $\sum_j y_j=0$ when we allow complex y_j. As we compute higher moments of the propagator momenta, the superficial degree of divergence of the Feynman amplitude increases until the amplitude needs to be regularised in order to converge; in terms of the generating function formalism, this implies that higher derivatives of the generating function are divergent at the origin. A way out offering itself almost naturally in formalisms embracing Schwinger parametrisation is dimensional regularisation (see e.g. [5, 1]).

Tree graphs. On tree graphs, one computes that the generating factor agrees precisely with the expected form

$$\exp -i\sum_{\text{prop. j}}y_j\cdot q_j.$$

There are no quadratic terms.

One-loop two-point function. This is the simplest non-trivial example. The diagram consists of a loop with two insertions, connected by two propagators which we will call left (1) and right (2). We choose as moduli the lengths τ_1 and τ_2 of the left and right branch. Rather than keeping two external momenta and imposing momentum conservation, we assume that there is one momentum k entering and leaving the loop. The momenta q_1 and q_2 are defined parallel to the canonical loop coordinate running around the loop and starting at the insertion where the external momentum k enters the loop. The dipole on the left branch is at coordinate s_1, on the right branch at coordinate s_2. The generating factor is then (with $T=\tau_1+\tau_2$)

$$e^{-\frac{i}{2}\left[(y_1\cdot k)\frac{T-2s_1}{T}+(y_2\cdot k)\frac{T-2s_2}{T}-(y_1\cdot k)\frac{T-2(s_1+\tau_2)}{T}-(y_2\cdot k)\frac{T-2(s_2-\tau_1)}{T}\right]-\frac{1}{4T}(y_1+y_2)^2}$$

$$=e^{-i\frac{(y_1\cdot k)\tau_2-(y_2\cdot k)\tau_1}{T}-\frac{1}{4T}(y_1+y_2)^2}.$$

We obtain the generating function (without coupling constants)

$$G_{\text{1-loop}}(k; y_1, y_2) = \frac{(2\pi)^d}{2} \int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2 \left(\frac{1}{4\pi(\tau_1 + \tau_2)} \right)^{d/2} e^{-m^2(\tau_1 + \tau_2)}$$
$$e^{-\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} k^2 - i \frac{y_1 \tau_2 - y_2 \tau_1}{\tau_1 + \tau_2} \cdot k - \frac{1}{4(\tau_1 + \tau_2)}(y_1 + y_2)^2}. \tag{5.6}$$

Relation to pre-integrated Schwinger representation. Of course, there is no magic in here. It is instructive to inspect the Fourier transform of the generating function $G_{\text{1-loop}}(k; y_1, y_2)$ with respect to the variables y_1, y_2. This will yield a kernel whose moments are equal to the derivatives of the generating function at $y_1 = y_2 = 0$; we have

$$(i\partial_1)^{n_1}(i\partial_2)^{n_2} G_{\text{1-loop}}(k; y_1, y_2)\big|_{y_1 = y_2 = 0}$$
$$= (2\pi)^{-2d} \int \mathrm{d}^d q_1 \, \mathrm{d}^d q_2 \, q_1^{n_1} q_2^{n_2} \int \mathrm{d}^d y_1 \, \mathrm{d}^d y_2 \, e^{iq_1 \cdot y_1 + iq_2 \cdot y_2} G_{\text{1-loop}}(k; y_1, y_2).$$

One finds with (5.6)

$$(2\pi)^{-2d} \int \mathrm{d}^d y_1 \, \mathrm{d}^d y_2 \, e^{iq_1 \cdot y_1 + iq_2 \cdot y_2} G_{\text{1-loop}}(k; y_1, y_2)$$
$$= \frac{1}{2} \delta^{(d)}(k + q_2 - q_1) \int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2 \, e^{-\tau_1(q_1^2 + m^2) - \tau_2(q_2^2 + m^2)}.$$

This is hardly a surprising result; the generating function is nothing more than the Fourier transform of the Schwinger parametrised amplitude, *before* integrating out the moments along the propagators. In the general case, this gives us a convenient way to obtain the generating functional.

6. Summary and outlook

We have demonstrated an "integrated" approach to the Schwinger parametrisation of connected Feynman amplitudes, encompassing the graph as an entity (world graph) by introducing specific boundary conditions at the vertices rather than breaking the graph at these points. With the necessary caveats of interpretation, it can be viewed as a diffusion process of splitting and re-fusing particles. The summation over Feynman graphs in order to obtain the total amplitude is thereby converted into a summation over equivalence classes of graphs. The Schwinger parameters are generalised to "Schwinger moduli" in the cells of moduli space corresponding to the equivalence classes. The vertices turn out to be by no means special points on the graph.

The prefactor $Z_0(\mathfrak{g}(\tau))$ providing the necessary normalisation is difficult to determine in general (although in specific cases, like the spider graph, it is clear). It would be desirable to find a closed formula for the determinant of the graph Laplacian, bringing us in a position to determine explicitly $Z_0(\mathfrak{g}(\tau))$.

Renormalisation is naturally included into this scheme by the assumption that the cancellation of infinities is local in moduli space. For this idea, the concept of a cell complex structure of moduli space is critical. As long as amplitudes are not renormalised

explicitly, the assumption of non-integer space dimensions is a solution adapted very well to the formalism.

The analogy to a system of charges on a graph which is a crucial step in the proof offers a simple way to include vector and tensor particles by way of a generating function formalism. The stress again lies not on the computational advantages of the scheme, but rather on the conceptual side: It is not necessary to sprinkle new terms over the path integral formula which do not have an intrinsic meaning. Rather, the generating function has a natural interpretation as containing test dipoles.

On the speculative side, we suggest the possibility of transforming by integration by parts the integral over the total moduli space cell complex associated to a correlation into a sum of integral contributions from minimal and maximal cells only. This might be of special interest in the treatment of non-Abelian gauge theories.

Looking further, the charge formalism carries the promise of a simple treatment of graphs containing particles carrying a "real world" electric charge. Such particles would bring along a cloud of photons coupling to the graph by a derivative coupling; the external photons would be represented by dipoles on the graph in the "charge formalism". Consequently, these dipoles would *shield* the pair potential along the world graph. In a similar vein, it might be advantageous to examine multi-particle production (which would presumably make the potential on the graph slightly random, like in a charged, grainy background medium).

The main motivation behind this work lies not in such "classical" issues, however; we hope that we might contribute to the rephrasing of the Schwinger parametrised perturbation amplitude into a bulk amplitude in the framework of the AdS/CFT correspondence.

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